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A Pseudodifferential Approach to Distributed Parameter Systems and Stabilization: Neumann Feedback Control Problems

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The recent developments in microlocal analysis and pdeudodifferential boundary calculus are well suited tools in the investigation of a large number of problems occurring in control theory for partial differential equations. We explain some of the basic ideas of a pseudodifferential model in the case of a distributed system with feedback acting on the boundary of a bounded domain in R^n and appearing in the Neumann boundary condition. We establish the pseudodifferental setting for the Neumann feedback control problem previously established for the corresponding Dirichlet problem by Pederson (*SIAM J. Control Optim.* **29** (1991)). The pseudodifferential techniques apply easily in the proof of existence of a feedback semigroup for the parabolic and hyperbolic evolution problems, and we reprove in this new setting some of the stabilization results of Lasiecka and Triggiani (see, e.g., *J. Differential Equations* **47** (1983); *Appl. Math. Optim.* **10** (1983)). So far, this work seems to have simplified or unified many of the previous works cited above. We hope that in the future it will even provide stronger and newer results in the boundary control of distributed parameter systems.

1. INTRODUCTION AND NOTATION

In this paper we establish the pseudodifferential setting for Neumann boundary feedback evolution problems previously established by the author for the corresponding Dirichlet problems in [P1]. We use (only a very small part of) the theory developed in [G1], it appears that the boundary feedback problems we consider are special cases of the very general pseudodifferential boundary problems considered there, and it is this viewpoint we are trying to promote.

We consider an open, bounded set $\Omega \subset R^n$, $n \geqslant 2$, with smooth boundary $\partial\Omega = \Gamma$, and a uniformly strongly elliptic differential operator A of order $2m$:

$$A = \sum_{|\alpha| \leqslant 2m} a_\alpha(x)\, D^\alpha, \tag{1.1}$$

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that is assumed to be formally selfadjoint, that is, selfadjoint when restricted to $C_0^\infty(\Omega)$, the space of test functions Here $\alpha=(\alpha_1, ..., \alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $D^\alpha=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$, where $D_j^{\alpha_j}=(-i\partial/\partial x_j)^{\alpha_j}$, and the coefficients $a_\alpha(x)$ are smooth. The classical Dirichlet and Neumann trace operators γ and υ are

$$\gamma=\{\gamma_j\}_{0\leqslant j<m}, \tag{1.2}$$

$$\upsilon=\{\gamma_j\}_{m\leqslant j<2m}, \tag{1.3}$$

where $\gamma_j u=(-i\partial/\partial n)^j u|_\Gamma$ (n denotes the normal).

To explain the idea of the pseudodifferential approach, we consider first the Neumann feedback problem for the (generalized) wave equation,

$$\partial_t^2 u+Au=0, \qquad (t,x)\in\,]0,\infty[\,\times\Omega, \tag{1.4}$$

with initial data $u(0,x)=u_0(x)$, $\partial_t u(0,x)=u_1(x)$, for $x\in\Omega$, at $t=0$. We write ∂_t for $\partial/\partial t$.

The boundary condition is

$$\upsilon u=T'u, \qquad (t,x)\in\,]0,\infty[\,\times\Gamma, \tag{1.5}$$

where T' is a feedback (trace) operator of a nature to be specified. The approach taken in the following is motivated by the study of controllability, stabilizability, and modelling of distributed systems for various choices of feedback operators T' and system operators A. Some interesting choices of T' occurring in the literature are, e.g.,

$$T'u=\sum_{j=1}^N\left(\int_\Omega uw_j\,d\Omega\right)g_j, \tag{1.6}$$

where the w_j are supported in Ω and the g_j are supported on Γ,

$$T'u=\sum_{j=1}^N\left(\int_\Gamma(\gamma u)\,w_j\,d\Gamma\right)g_j, \tag{1.7}$$

where both the w_j and the g_j are supported on Γ. One could also replace γ with υ in (1.7). We refer to (among others) [LT1, LT2, LT3, LT4, LT5, LT6, C1, N1, T1, LLT1, L1].

From our point of view, the significant fact is that (assuming sufficient regularity of the functions), all the choices of T' above define so-called normal trace operators in the pseudodifferential boundary calculus, see [G1] for the precise definitions and [P1, P2, P3, P4] for applications, giving rise to normal boundary value problems.

2. Pseudodifferential Transformations

Denote by A_υ the operator that acts as A does in $L^2(\Omega)$, and with the domain

$$D(A_\upsilon) = \{u \in H^{2m}(\Omega) \mid \upsilon u = 0\}, \tag{2.1}$$

where $H^s(\Omega)$ is the Sobolev space of distributions on Ω with $L^2(\Omega)$-derivatives up to order s. Let us denote by A_1 the operator that acts as A does in $L^2(\Omega)$, but with the domain

$$D(A_1) = \{u \in H^{2m}(\Omega) \mid \upsilon u = T'u\}, \tag{2.2}$$

where T' denotes a normal trace operator of the type mentioned above (see also below). Then there exists an isomorphism and homeomorphism Λ in $H^s(\Omega)$, for all $s \geqslant 0$, such that

$$\Lambda : D(A_1) \to D(A_\upsilon), \tag{2.3}$$

and such that $\Lambda \partial_t - \partial_t \Lambda = 0$, with $\Lambda\Lambda = \Lambda$ in $L^2(\Omega)$, see [P1, G1, Lemma 1.6.8]. The most important difference between the boundary conditions $\upsilon u = 0$ and $\upsilon u - T'u = 0$ is that the first one is *local*, that is, acts only locally on Γ, whereas the second one typically will be *non-local*. The transformation operator Λ is of pseudodifferential nature that will be explained briefly below. Acting with Λ on the hyperbolic boundary feedback system

$$\partial_t^2 u + A_1 u = 0, \qquad u \in D(A_1), \tag{2.4}$$

gives, with $v = \Lambda u$, that

$$\partial_t^2 \Lambda^{-1} v + A_1 \Lambda^{-1} v, \qquad v \in D(A_\upsilon), \tag{2.5}$$

and, applying Λ from the left, we arrive at

$$\partial_t^2 v + \Lambda A_1 \Lambda^{-1} v = 0, \qquad v \in D(A_\upsilon), \tag{2.6}$$

Now, by construction

$$A_\upsilon = A_1 \Lambda^{-1}, \tag{2.7}$$

so the original nonlocal boundary feedback system is transformed into

$$\partial_t^2 v + \Lambda A_\upsilon v = 0, \qquad v \in D(A_\upsilon), \tag{2.8}$$

where the boundary condition now is local (at the expense of nonlocal terms in the operator equation).

This approach works in exactly the same way for the parabolic problem

$$\partial_t u + A_1 u = 0, \qquad (t, x) \in]0, \infty[\times \Omega, \tag{2.9}$$

for $u \in D(A_1)$ and with initial data $u(0, x) = u_0(x)$, for $t = 0$ and $x \in \Omega$. See Section 5 below.

3. Structure Analysis

To understand the action of the Λ transformation, we must briefly introduce some of the ingredients entering in the pseudodifferential calculus of boundary problems. For the most detailed and comprehensive study of this theory we refer to [G1].

The pseudodifferential operator P on R^n is defined by

$$Pu(x) = (2\pi)^{-n} \int_{R^n} e^{ix\xi} p(x, \xi)\, \hat{u}(\xi)\, d\xi, \tag{3.1}$$

where $\hat{u}(\xi) = \mathscr{F}_{x \to \xi} u(\xi)$ denotes the Fourier transform of u. The (smooth) function p is the *symbol* of P, and when, for some $d \in R$ and all multiindices $\alpha, \beta \in N^n$, there is a continuous function $c_{\alpha,\beta}$, such that

$$|D_x^\beta D_\xi^\alpha p(x, \xi)| \leqslant c_{\alpha,\beta}(x)(1 + |\xi|^2)^{(d - |\alpha|)/2}, \tag{3.2}$$

we say that P is of order d (and p belongs to the symbol class $S^d_{1,0}$), corresponding to the case where P is a differential operator and p is a polynomial in ξ. Then the restriction of P to functions on Ω is defined by

$$P_\Omega u = r_\Omega P e_\Omega u, \tag{3.3}$$

where r_Ω is the restriction operator from $\mathscr{D}'(R^n)$ to $\mathscr{D}'(\Omega)$ (spaces of distributions on R^n, resp. Ω), and e_Ω is the "extension by zero"-operator, extending functions on Ω to R^n by setting them equal to 0 outside Ω.

But other types of operators enter naturally in the discussion of elliptic boundary value problems. The most simple example is perhaps the Helmholtz equation (see [G2]):

$$\begin{aligned} (1 - \Delta) u &= f, \qquad \text{in } \Omega, \\ \gamma u &= \varphi \qquad \text{on } \Gamma, \end{aligned} \tag{3.4}$$

where we take $\Omega = R^n_+$. Writing $x = (x', x_n)$, $\xi = (\xi', \xi_n)$, the solution is

$$u = R_\gamma f + K_\gamma \varphi, \tag{3.5}$$

where K_γ is a *Poisson operator*,

$$K_\gamma\,\varphi(x) = (2\pi)^{1-n} \int_{R^{n-1}} e^{ix'\xi'}\, e^{-(1+|\xi'|^2)^{1/2}x_n}\, \hat{\varphi}(\xi')\, d\xi', \tag{3.6}$$

and

$$R_\gamma = Q_\Omega + G_\gamma, \tag{3.7}$$

where Q_Ω is defined as above from the pseudodifferential operator with symbol $(1+|\xi|^2)^{-1}$. G_γ is a *singular Green operator*,

$$G_\gamma f(x) = (2\pi)^{1-n} \int_{R^{n-1}} e^{ix'\xi'}(-2(1+|\xi'|^2)^{1/2})^{-1}$$
$$\times \int_0^\infty e^{-(1+|\xi'|^2)^{1/2}(x_n+y_n)}(\mathscr{F}_{y'\to\xi'} f)(\xi', y_n)\, dy_n\, d\xi'.$$

Here $\mathscr{F}_{y'\to\xi'} f$ denotes the partial Fourier transform in y' of $f(y', y_n)$. Thus we see that operators (besides the standard pseudodifferential) that necessarily must be considered when discussing boundary value problems are:

trace operators, like γ, v, and T', mapping functions supported in Ω into functions supported on Γ,

Poisson operators, like K_γ, mapping functions supported on Γ into functions supported in Ω, and

singular Green operators, like G_γ, mapping functions supported in Ω into functions supported in Ω.

An important fact from the calculus is that the composition KT of a Poisson operator K and a trace operator T is a singular Green operator. In general, the transformation operator Λ can be chosen such that

$$\Lambda = 1 - K_0 T' \tag{3.9}$$

and

$$\Lambda^{-1} = 1 - K_0 Q_0 T', \tag{3.10}$$

where K_0 is a standard type of Poisson operator, chosen such that $K_0 T'$ has small norm, and Q_0 is a certain pseudodifferential operator that is bijective and elliptic in $\prod_{0\leqslant k<2m} H^{s-k}(\Gamma)$, for $s \geqslant 0$. See [G1, Lemma 1.6.8].

Let us proceed with this very short presentation of the symbolic calculus and give the precise estimates for the trace, Poisson, and singular green operators (compare with (3.6) and (3.8)). These are vital for the study of, e.g., the regularity of feedback solutions to control problems; see also Remark 4.2 below.

A *trace operator* of order d (and class r) is of the form

$$T = \sum_{0 \leqslant j < r} S_j \gamma_j + T'_0, \tag{3.11}$$

where the S_j are pseudodifferential operators on Γ of order $d-j$ and T'_0 has the form (in halfspace coordinates)

$$T'_0 u(x') = (2\pi)^{-1-n} \int_{R^{n-1}} e^{ix'\xi'} \times \int_0^\infty t'_0(x', x_n, \xi') \mathscr{F}_{x' \to \xi'} u(\xi', x_n)\, dx_n\, d\xi', \tag{3.12}$$

where t'_0 is a rapidly decreasing function of the x_n-variable, satisfying the estimate

$$\|x_n^l D_{x_n}^{l'} D_{x'}^{\alpha'} D_{\xi'}^{\alpha'} t'_0(x', x_n, \xi')\|_{L^2_{x_n}(R_+)} \leqslant c(x')(1 + |\xi'|^2)^{(d + 1/2 - l + l' - |\alpha'|)/2}, \tag{3.13}$$

for multiindices α' and β' in N^{n-1}.

Corresponding to this, a Poisson operator K of order d can be written

$$K\varphi(x', x_n) = (2\pi)^{1-n} \int_{R^{n-1}} e^{ix'\xi'} k(x', x_n, \xi')\, \hat{\varphi}(\xi')\, d\xi', \tag{3.14}$$

where the kernel k satisfies

$$\|x_n^l D_{x_n}^{l'} D_{x'}^{\beta'} D_{\xi'}^{\alpha'}\|_{L^2_{x_n}(R_+)} \leqslant c(x')(1 + |\xi'|^2)^{(d - 1 + 1/2 - l + l' - |\alpha'|)/2}. \tag{3.15}$$

One very important fact is that the singular Green operators of order d can be written

$$G = \sum_{0 \leqslant j < r} K_j \gamma_j + G', \tag{3.16}$$

where the K_j are Poisson operators of order $d-j$, and G' is an operator of the form

$$G'u(x) = (2\pi)^{1-n} \int_{R^{n-1}} e^{ix'\xi'} \times \int_0^\infty g'(x', x_n, y_n, \xi') \mathscr{F}_{x' \to \xi'}(\xi', y_n)\, dy_n\, d\xi'. \tag{3.17}$$

Here g' is a rapidly decreasing function of x_n and y_n in $\bar{R}_+$, satisfying the estimate:

$$\|x_n^k D_{x_n}^{k'} y_n^m D_{y_n}^{m'} D_{x'}^{\beta'} D_{\xi'}^{\alpha'} g'(x', x_n, y_n, \xi')\|_{L^2_{x_n, y_n}(\bar{R}_+ \times \bar{R}_+)}$$
$$\leqslant c(x')(1+|\xi'|^2)^{(d-k+k'-m+m'-|\alpha'|)/2}. \tag{3.18}$$

By expansion of g' in a Laguerre series, we see that actually

$$G' = \sum_{m=1}^{\infty} K_m T_m, \tag{3.19}$$

where the terms of the series (3.19) are rapidly decreasing with respect to all the symbol norms. The operator G' is compact in $L^2(\Omega)$, hence all singular Green operators can be written in the form (3.16), as a combination of classical trace operators composed with Poisson operators + a term that is compact in $L^2(\Omega)$.

4. A Simple Transformation

There is a transformation operator Λ particularly adapted to our Neumann problem. Let T' be a trace operator of the form (1.6) and define Λ as

$$\Lambda = 1 - K_\nu T', \tag{4.1}$$

where K_ν now denotes the solution operator to the Neumann problem for A; i.e., $u = K_\nu \varphi$, where $Au = 0$ in Ω and $\nu u = \varphi$ on Γ. Then, for $u \in D(A_1)$, we have that $v = \Lambda u$ satisfies

$$Av = A(1 - K_\nu T')u = Au, \tag{4.2}$$

and, moreover,

$$\nu v = \nu(1 - K_\nu T')u = \nu u - T'u = 0; \tag{4.3}$$

hence $v \in D(A_\nu)$.

Now (2.8) reads

$$\partial_t^2 v + (1 - K_\nu T')\, A_\nu v = 0, \qquad v \in D(A_\nu), \tag{4.4}$$

and the operator $K_\nu T'A$ is a standard singular Green operator.

For the Λ-transformation (2.3), (4.1) to be an isomorphism and homeomorphism, the set $\{w_j, g_j\}$, $j = 1, 2, ..., N$, must be chosen in an appropriate way that will be indicated below. We will also carry out all calculations modulo the constant functions. We have shown

THEOREM 4.1. *The feedback problems*

$$\begin{aligned} \partial_t^2 u + Au &= 0, & x\in\Omega,\quad & t>0,\\ \upsilon u &= T'u & x\in\Gamma,\quad & t>0,\\ u(0) &= u_0(x), & x\in\Omega,\quad & t=0,\\ \partial_t u(0) &= u_1(x), & x\in\Omega,\quad & t=0, \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \partial_t u + Au &= 0, & x\in\Omega,\quad & t>0,\\ \upsilon u &= T'u & x\in\Gamma,\quad & t>0,\\ u(0) &= u_0(x), & x\in\Omega,\quad & t=0, \end{aligned} \tag{4.6}$$

can, for sufficiently smooth data, be transformed into the systems

$$\begin{aligned} \partial_t^2 v + Av - K_\upsilon T'Av &= 0, & x\in\Omega,\quad & t>0,\\ \upsilon v &= 0, & x\in\Gamma,\quad & t>0,\\ v(0) &= v_0(x), & x\in\Omega,\quad & t=0,\\ \partial_t v(0) &= v_1(x), & x\in\Omega,\quad & t=0, \end{aligned}$$

respectively,

$$\begin{aligned} \partial_t v + Av - K_\upsilon T'Av &= 0, & x\in\Omega,\quad & t>0,\\ \upsilon v &= 0, & x\in\Gamma,\quad & t>0,\\ v(0) &= v_0(x), & x\in\Omega,\quad & t=0. \end{aligned}$$

Remark 4.2. In view of well-known facts concerning the stabilization of distributed systems, the stabilizing effect of the boundary feedback operator T' can be investigated by investigation of the singular Green operator $K_\upsilon T'A$, where the compact contributions can be neglected when *hyperbolic* systems are considered. For *parabolic* systems the situations is different, see [P1] and below.

From the formulation of Theorem 4.1, it is obvious that the operator A_1 is the infinitesimal generator of an analytic semigroup $e^{-A_1 t}$, because A_υ is, and $K_\upsilon T'A$ is A-bounded. Note the identity

$$e^{-A_1 t} = (1-K_\upsilon T')^{-1}\, e^{-(A_1 - K_\upsilon T'A_1)t}(1-K_\upsilon T'), \tag{4.7}$$

for the semigroups. The result for the sine and cosine operators are similar, we refer to [P1] for more details. Now we will proceed with a stability analysis, this turns out to be quite easy, essentially because of (4.7).

5. An Application of the Transformation to Stabilization

We will now show how the pseudodifferential transformation Λ can be used to prove some of the stabilization results of Lasiecka and Triggiani in, e.g., [LT2, LT3, LT4]; see also Remark 4.4 in [G3], in a manner similar to the one used for the Dirichlet problems in [P1].

It is well known how it is the negative eigenvalues in the pure point spectrum of A_υ that causes the system $\partial_t u + Au = 0$, with boundary condition $\upsilon u = 0$ and some initial data in $L^2(\Omega)$, to be exponentially unstable. It is also well known that there are only finitely many negative eigenvalues, so let the eigenvalues of A_υ be arranged in a non-decreasing sequence

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{K-1} < 0 < \lambda_K \leqslant \cdots, \tag{5.1}$$

each eigenvalue repeated according to multiplicity. Let $\{\varphi_j\}_{j \geqslant 1}$ be a corresponding set of orthonormalized eigenfunctions for A_υ. Now define P_u and P_s as the orthogonal projections of $L^2(\Omega)$ on the orthogonal subspaces X_u, respectively X_s, defined by

$$\begin{aligned} X_u &= \{\varphi_j\}_{1 \leqslant j < K}, \\ X_s &= \overline{\operatorname{span}}\{\varphi_j\}_{j \geqslant K}. \end{aligned} \tag{5.2}$$

Since X_u and $X_s \cap D(A_\upsilon)$ are invariant subspaces for A_υ, we can define the restrictions

$$\begin{aligned} A_u &= A_\upsilon|_{X_u}, \\ A_s &= A_\upsilon|_{X_s \cap D(A_\upsilon)}. \end{aligned} \tag{5.3}$$

Then A_u is a bounded operator on X_u and A_s is an unbounded operator with domain $D(A_s) = X_s \cap D(A_\upsilon)$. Note that P_u and P_s commute with A_υ on $D(A_\upsilon)$. Now writing $f_u = P_u f$, $f_s = P_s f$ for $f \in L^2(\Omega)$, we have that when $u \in D(A_1)$, then $v = (1 - K_\upsilon T')u \in D(A_\upsilon)$ satisfies

$$\begin{aligned} &Av = Au, \\ &v_u \in X_u, \\ &v_s \in D(A_s). \end{aligned} \tag{5.4}$$

We use the factorization

$$A_1 = A_\upsilon(1 - K_\upsilon T') \tag{5.5}$$

in the discussion of the resolvent equation

$$(A_1 - \lambda)u = f, \qquad f \in L^2(\Omega). \tag{5.6}$$

First we consider the case where we are allowed to *decouple* the feedback by assuming that

$$P_s w_j = 0, \qquad j = 1, 2, ..., N \tag{5.7}$$

(i.e., the w_j are in X_u; the "unstable" eigenspace).

Then we write (5.6) in projected and factorized form

$$\begin{pmatrix} P_u \\ P_s \end{pmatrix} (A_v(1 - K_v T')(u_u + u_s) - \lambda(u_u + u_s)) = \begin{pmatrix} f_u \\ f_s \end{pmatrix} \tag{5.8}$$

and we compute, for $u \in D(A_1)$,

$$\begin{aligned} &P_u A_v(1 - K_v T')(u_u + u_s) - P_u \lambda(u_u + u_s) \\ &\quad = A_v P_u(1 - K_v T')(u_u + u_s) - \lambda u_u \\ &\quad = A_u u_u - A_u P_u K_v T' u_u - \lambda u_u, \\ &P_s A_v(1 - K_v T')(u_u + u_s) - P_s \lambda(u_u + u_s) \\ &\quad = A_v P_s(1 - K_v T')(u_u + u_s) - \lambda u_s \\ &\quad = A_s(u_s - P_s K_v T' u_u) - \lambda u_s. \end{aligned}$$

Since $u_s - P_s K_v T' u_u = v_s$ belongs to $D(A_s)$, the factorization (5.8) is legitime, and (5.8) reduces to

$$A_u u_u - A_u P_u K_v T' u_u - \lambda u_u = f_u, \tag{5.9}$$

$$A_s(u_s - P_s K_v T' u_u) - \lambda u_s = f_s, \tag{5.10}$$

where we observe that (5.9) is a finite-dimensional resolvent equation for the matrix operator

$$\bar{A}_u = A_u - A_u P_u K_v T'. \tag{5.11}$$

At this point we can use the same arguments as Lasiecka and Triggiani in, e.g., [LT2] to get a good choice of T'. We give the full details in the most straightforward case and refer to Lasiecka and Triggiani for partial informations on other cases. First we will formulate Greens formulas for the operator A in a convenient way. We denote by $(\cdot|\cdot)$ and $(\cdot|\cdot)_\Gamma$ the inner products in $L^2(\Omega)$, resp. $L^2(\Gamma)^m$. Using this notation we have, for the formally selfadjoint operator A that

$$(Au | v) - (u | Av) = (\mathscr{A} \rho u | \rho v)_\Gamma, \tag{5.12}$$

where $\mathscr{A}$ is a skew-triangular $2m \times 2m$-matrix of differential operators over the boundary Γ, of the form

$$\mathscr{A} = \begin{pmatrix} S_1^0 & \cdots & S_{2m-1}^0 & S_{2m}^0 \\ S_2^0 & \cdots & S_{2m}^0 & 0 \\ \vdots & & & \\ S_{2m}^0 & \cdots & 0 & 0 \end{pmatrix} + \begin{pmatrix} \text{lower} & & 0 \\ \text{order} & & \\ & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}, \tag{5.13}$$

the S_k^0 being differential operators on Γ of order $2m-k$ (see, e.g., Grubb [G1, Section 1.3]).

We usually write $\mathscr{A}$ in $m \times m$ blocks as

$$\mathscr{A} = \begin{pmatrix} \mathscr{A}^{00} & \mathscr{A}^{01} \\ \mathscr{A}^{10} & 0 \end{pmatrix} \tag{5.14}$$

and we have the following version of Greens formula

$$(Au \mid v) - (u \mid Av) = (\mathscr{A}^{01} \nu u + \mathscr{A}^{00} \gamma u \mid \gamma v)_\Gamma + (\mathscr{A}^{10} \gamma u \mid \nu v)_\Gamma \tag{5.15}$$

for $u, v \in H^{2m}(\Omega)$. The coefficient matrices $\mathscr{A}^{ij}$ here are uniquely determined from A; $\mathscr{A}^{01}$ and $\mathscr{A}^{10}$ are invertible since A is elliptic. Inserting $u = \varphi_j$ and $v = K_\nu \psi$ in 5.15 we obtain the formula

$$(K_\nu \psi \mid \varphi_j) = \frac{1}{\lambda_j} (\psi \mid (K_\nu^* \gamma^* \mathscr{A}^{00} + \mathscr{A}^{10}) \gamma \varphi_j)_\Gamma, \tag{5.16}$$

which we will use below.

THEOREM 5.1. *Assume that the Dirichlet traces* $\{\gamma\varphi_j\}_{1 \leqslant j < K}$ *are linearly independent, so that*

$$\dim(\gamma X_u) = \dim(X_u) \qquad (= K-1), \tag{5.17}$$

and let $\{c_j\}_{1 \leqslant j < K}$ *be an arbitrary given set of* $K-1$ *distinct, real numbers.*

Then there exists a number N *and a set*

$$\{w_j, g_j\}_{1 \leqslant j \leqslant N},$$

where $w_j \in X_u$ *and* $g_1 \in C^\infty(\Gamma)^m$, *such that with boundary condition* $\nu u = T'u$, *where*

$$T'u = \sum_{j=1}^{N} (u \mid w_j)\, g_j, \tag{5.18}$$

the eigenvalues of the matrix operator $\bar{A}_u$, *defined by*

$$\bar{A}_u v = (A_u - A_u P_u K_\nu T')v \qquad \textit{for} \quad v \in X_u, \tag{5.19}$$

are precisely the set $\{c_j\}_{1 \leqslant j < K}$. *The number* N *can be taken as the largest multiplicity of the unstable eigenvalues* $\{\lambda_j\}_{1 \leqslant j < K}$. *In particular,* $N = 1$ *when the eigenvalues are simple.*

Proof. Assume first that *all* of the eigenvalues $\{\lambda_j\}_{1 \leqslant j < K}$ are simple and take $N = 1$. Consider T' of the form

$$T'u = (u \mid w)\, g. \tag{5.20}$$

In the basis $\{\varphi_j\}_{1 \leqslant j < K}$ of X_u, the matrix $\bar{A}_u$ has the form

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \lambda_{K-1} \end{pmatrix} - P(g)\, W^t, \tag{5.21}$$

where $P(g)$ is the column vector

$$P(g) = \begin{pmatrix} \lambda_1 (K_v g \mid \varphi_1) \\ \lambda_2 (K_v g \mid \varphi_2) \\ \vdots \\ \lambda_{K-1} (K_v g \mid \varphi_{K-1}) \end{pmatrix} \tag{5.22}$$

and W^t is the transpose of the column vector W given by

$$W = \begin{pmatrix} (\varphi_1 \mid w) \\ (\varphi_2 \mid w) \\ \vdots \\ (\varphi_{K-1} \mid w) \end{pmatrix}. \tag{5.23}$$

Consider now the control matrix of the pair (A_u, W):

$$\begin{aligned} &[W A_u W \cdots A_u^{K-2} W] \\ &= \begin{pmatrix} (\varphi_1 \mid w) & \lambda_1 (\varphi_1 \mid w) & \cdots & \lambda_1^{K-2} (\varphi_1 \mid w) \\ (\varphi_2 \mid w) & \lambda_2 (\varphi_2 \mid w) & \cdots & \lambda_2^{K-2} (\varphi_2 \mid w) \\ \vdots & & & \\ (\varphi_{K-1} \mid w) & \lambda_{K-1} (\varphi_{K-1} \mid w) & \cdots & \lambda_{K-1}^{K-2} (\varphi_{K-1} \mid w) \end{pmatrix}. \end{aligned} \tag{5.24}$$

The determinant of the control matrix is calculated to be

$$\prod_{1 \leqslant j < K} (\varphi_j \mid w) \cdot \prod_{1 \leqslant l < l < K} (\lambda_k - \lambda_l) \tag{5.25}$$

by reduction to a Vandermonde determinant. Since the eigenvalues are assumed to be simple, we can choose $w \in X_u$, satisfying $(\varphi_j \mid w) \neq 0$ for

$j = 1, 2, \dots, K-1$, such that the determinant is different from 0. This implies that the pair (A_u, W) is controllable, so by the pole assignment theorem (Wonham [W1]) there exists a matrix

$$\tilde{P} = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_{K-1} \end{pmatrix}, \qquad p_j \in \mathbb{C},\ j = 1, 2, \dots, K-1, \tag{5.26}$$

for which the matrix $A_u - \tilde{P}W^t$ has the set $\{c_j\}_{1 \leqslant j < K}$ as eigenvalues. Now we will choose $g \in C^\infty(\Gamma)^m$ such that $\tilde{P} = P(g)$, i.e., such that

$$\lambda_j (K_\upsilon g \mid \varphi_j) = p_j, \tag{5.27}$$

for $j = 1, 2, \dots, K-1$.

From the formula (5.16) we have that

$$(K_\upsilon g \mid \varphi_j) = \frac{1}{\lambda_j} (g \mid (K_\upsilon^* \gamma^* \mathscr{A}^{00} + \mathscr{A}^{10}) \gamma \varphi_j)_\Gamma, \tag{5.28}$$

Here $\mathscr{A}^{10}$ is an (since A is elliptic) invertible $m \times m$ matrix of differential operators over Γ. Since the set

$$\{\gamma\varphi_1, \gamma\varphi_2, \dots, \gamma\varphi_{K-1}\}, \tag{5.29}$$

is linearly independent, so also is the set

$$\{(K_\upsilon^* \gamma^* \mathscr{A}^{00} + \mathscr{A}^{10}) \gamma \varphi_j\}_{1 \leqslant j < K}. \tag{5.30}$$

Hence it is possible to choose $g \in C^\infty(\Gamma)^m$, satisfying

$$(g \mid (K_\upsilon^* \gamma^* \mathscr{A}^{00} + \mathscr{A}^{10}) \gamma \varphi_j)_\Gamma = p_j, \qquad j = 1, 2, \dots, K-1, \tag{5.31}$$

and this choice of g gives us the components of the desired $P(g)$, in view of (5.28). This ends the proof in the case of simple eigenvalues.

We now assume that one or more of the eigenvalues $\{\lambda_j\}_{1 \leqslant j < K}$ have multiplicity larger than one, and let us take σ to be the largest occurring multiplicity. Take $N = \sigma$ and consider T' of the form

$$T'u = \sum_{j=1}^{\sigma} (u \mid w_j)\, g_j. \tag{5.32}$$

In this case $A_u - A_u P_u K_\upsilon T'$ can be written

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & \lambda_{K-1} \end{pmatrix} - P(\{g_i\}_{1 \leqslant i \leqslant \sigma})\, W'_\sigma, \tag{5.33}$$

where $P(\{g_i\}_{1\leqslant i\leqslant\sigma})$ is the $(K-1)\times\sigma$-matrix

$$P(\{g_i\}_{1\leqslant i\leqslant\sigma})$$
$$=\begin{pmatrix} \lambda_1(K_v g_1\,|\,\varphi_1) & \lambda_1(K_v g_2\,|\,\varphi_2) & \cdots & \lambda_1(K_v g_\sigma\,|\,\varphi_1) \\ \lambda_2(K_v g_1\,|\,\varphi_2) & \lambda_2(K_v g_2\,|\,\varphi_2) & \cdots & \lambda_2(K_v g_\sigma\,|\,\varphi_2) \\ \vdots & \vdots & & \\ \lambda_{K-1}(K_v g_1\,|\,\varphi_{K-1}) & \lambda_{K-1}(K_v g_2\,|\,\varphi_{K-1}) & \cdots & \lambda_{K-1}(K_v g_\sigma\,|\,\varphi_{K-1}) \end{pmatrix} \tag{5.34}$$

and W_σ is the $(K-1)\times\sigma$-matrix

$$W_\sigma=\begin{pmatrix} (\varphi_1\,|\,w_1) & (\varphi_1\,|\,w_2) & \cdots & (\varphi_1\,|\,w_\sigma) \\ (\varphi_2\,|\,w_1) & (\varphi_2\,|\,w_2) & \cdots & (\varphi_2\,|\,w_\sigma) \\ \vdots & & & \\ (\varphi_{K-1}\,|\,w_1) & (\varphi_{K-1}\,|\,w_2) & \cdots & (\varphi_{K-1}\,|\,w_\sigma) \end{pmatrix}. \tag{5.35}$$

Considering the form of the control matrix

$$[W_\sigma A_u W_\sigma \cdots A_u^{K-2} W_\sigma], \tag{5.36}$$

we see that if $w_1, w_2, ..., w_\sigma$ are chosen in X_u such that

$$\operatorname{rank} W_\sigma=\sigma, \tag{5.37}$$

then the rank of the control matrix (5.36) is $K-1$ (because a regular $(K-1)\times(K-1)$-submatrix can be extracted, after suitable row–column operations). Then according to Wonham's theorem there exists a complex matrix

$$\tilde{P}_\sigma=\begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1\sigma} \\ p_{21} & p_{22} & \cdots & p_{2\sigma} \\ \vdots & \vdots & \vdots & \\ p_{K-1,1} & p_{K-1,2} & \cdots & p_{K-1,\sigma} \end{pmatrix} \tag{5.38}$$

such that the eigenvalues of the matrix $A_u-\tilde{P}_\sigma W'_\sigma$ are $\{c_j\}_{1\leqslant j<K}$.

To obtain $\tilde{P}_\sigma=P(\{g_i\}_{1\leqslant i\leqslant\sigma})$ we use that, as in (5.28),

$$(K_v g_i\,|\,\varphi_j)=\frac{1}{\lambda_j}(g_i\,|\,(K_v^*\gamma^*\mathscr{A}^{00}+\mathscr{A}^{10})\,\gamma\varphi_j)_\Gamma. \tag{5.39}$$

In view of (5.17) we can choose $g_i\in C^\infty(\Gamma)^m$, $i=1, 2, ..., \sigma$, satisfying

$$(g_i\,|\,(K_v^*\gamma^*\mathscr{A}^{00}+\mathscr{A}^{10})\,\gamma\varphi_j)_\Gamma=p_{ji}, \quad j=1, 2, ..., K-1; i=1, 2, ..., \sigma, \tag{5.40}$$

and this choice of $\{g_i\}_{1\leqslant i\leqslant\sigma}$ provides the desired $P(\{g_i\}_{1\leqslant i\leqslant\sigma})$.

Remark 5.2. For the application of the Wonham theorem here, it is important that the range of $P_u K_v$ fills out all of X_u; this is reformulated as the question of whether the Dirichlet traces of the Neumann eigenfunctions in X_u are linearly independent. In that case the results are easy to formulate and allow N to be very low; otherwise the results become increasingly complicated and require (in general) higher N, the more linear dependence there is. Lasiecka and Triggiani have in [LT6] an upper and lower bound on the number N of feedback terms necessary in (5.18), once the number of linearly independent Dirichlet traces are given, but the discussion of the size of $\dim(\gamma X_u)$ for differential operators on general domains in $\mathbb{R}^n$ in the literature is far from complete, as far as we know.

Let us now choose the set $\{c_j\}_{1 \leqslant j < K}$ occurring in Theorem 5.1 such that $c_j \geqslant \lambda_K (>0)$, $j = 1, 2, ..., K-1$.

With T' chosen according to the theorem, the operator $P_u(1 - K_v T')$ is injective, hence bijective, on X_u, and since $w_j \in X_u$, $1 - K_v T'$ is the identity on X_s. Then, as promised earlier, $1 - K_v T'$ is bijective from $H^{2m}(\Omega)$ to $H^{2m}(\Omega)$ ($K_v T'$ has C^∞-range) and maps $D(A_1)$ onto $D(A_v)$. Thus justifies the factorization (5.8).

Define $R(\lambda, \bar{A}_u)$ as the resolvent of $\bar{A}_u$ in X_u, for all $\lambda \notin \{c_j\}_{1 \leqslant j < K}$ and let $R(\lambda, A_s)$ be the resolvent of A_s in X_s, defined for all $\lambda \notin \{\lambda_j\}_{j \geqslant K}$.

We can then write the solution to (5.9) as

$$u_u = R(\lambda, \bar{A}_u) f_u, \qquad \lambda \notin \{c_j\}_{1 \leqslant j < K}. \tag{5.41}$$

We use that if $u \in D(A_1)$, then $v = (1 - K_v T')u$ belongs to $D(A_v)$; hence $v_s \in D(A_s)$. Since

$$\begin{aligned} v_s &= P_s(1 - K_v T')u = P_s(1 - K_v T')(u_u + u_s) \\ &= u_s - P_s K_v T' u_u, \end{aligned} \tag{5.42}$$

we see that $u_s - P_s K_v T' u_u \in D(A_s)$, so (5.10) is justified and can be written as

$$(A_s - \lambda)(u_s - P_s K_v T' u_u) = f_s + \lambda P_s K_v T' u_u. \tag{5.43}$$

For all $\lambda \notin \{\lambda_j\}_{j \geqslant K}$ we have

$$u_s - P_s K_v T' u_u = R(\lambda, A_s)(f_s + \lambda P_s K_v T' u_u). \tag{5.44}$$

Inserting (5.41), we find for all $\lambda \notin (\{c_j\}_{1 \leqslant j < K} \cup \{\lambda_j\}_{j \geqslant K})$,

$$u_s = P_s K_v T' R(\lambda, \bar{A}_u) f_u + R(\lambda, A_s)(f_s + \lambda P_s K_v T' R(\lambda, \bar{A}_u) f_u), \tag{5.45}$$

so that

$$
\begin{aligned}
u &= u_u + u_s \\
&= R(\lambda, \bar{A}_u) f_u + P_s K_\upsilon T' R(\lambda, \bar{A}_u) f_u + R(\lambda, A_s) f_s \\
&\quad + \lambda R(\lambda, A_s) P_s K_\upsilon T' R(\lambda, \bar{A}_u) f_u \\
&= (1 + P_s K_\upsilon T' + \lambda R(\lambda, A_s) P_s K_\upsilon T') R(\lambda, \bar{A}_u) f_u + R(\lambda, A_s) f_s. \qquad (5.46)
\end{aligned}
$$

We have now obtained

THEOREM 5.3. *The resolvent $R(\lambda, A_1)$ solving* (5.9)–(5.10) *can be written in the form*

$$R(\lambda, A_1) f = (T_{11} R_{12}) \begin{pmatrix} f_u \\ f_s \end{pmatrix}, \qquad (5.47)$$

where

$$
\begin{aligned}
R_{11} &= (1 + P_s K_\upsilon T' + \lambda R(\lambda, A_s) P_s K_\upsilon T') R(\lambda, \bar{A}_u), \\
R_{12} &= R(\lambda, A_s),
\end{aligned} \qquad (5.48)
$$

so $R(\lambda, A_1)$ is well defined for all λ outside the spectrum of A_1 and maps $L^2(\Omega)$ into $H^{2m}(\Omega)$.

Let us prove

THEOREM 5.4 *Assume that the prescribed eigenvalues $\{c_j\}_{1 \leqslant j < K}$ of $\bar{A}_u$ are chosen such that $c_j \geqslant \lambda_K$, $j = 1, ..., K-1$. There is then a constant $M_1 > 0$, independent of λ, such that the resolvent $R(\lambda, A_1)$ satisfies the inequality*

$$\|R(\lambda, A_1)\|_{L^2, L^2} \leqslant \frac{M_1}{\operatorname{dist}(\lambda, I_K)} \qquad (5.49)$$

as an operator in $L^2(\Omega)$. Here $I_K = [\lambda_K, \infty[$, which in this case equals the closed convex hull of the spectrum of A_1.

Proof. Let $\|\cdot\|$ denote the $L^2(\Omega)$-norm and $\|\cdot\|_{L^2, L^2}$ the corresponding operator norm. A_s is a self-adjoint, positive operator on $X_s \cap D(\upsilon)$, satisfying

$$
\begin{aligned}
\|(A_s - \lambda) u\| \, \|u\| &\geqslant |((A_s - \lambda) u \mid u)| \\
&= |((A_s - \operatorname{Re} \lambda) u \mid u) - i \operatorname{Im} \lambda \|u\|^2| \\
&= (((A_s - \operatorname{Re} \lambda) u \mid u)^2 + (\operatorname{Im} \lambda \|u\|)^2)^{1/2} \\
&\geqslant \begin{cases} |\operatorname{Im} \lambda| \, \|u\|^2 & \text{if} \quad \operatorname{Re} \lambda \geqslant \lambda_K \\ ((\lambda_K - \operatorname{Re} \lambda)^2 + (\operatorname{Im} \lambda)^2)^{1/2} \|u\|^2 & \text{if} \quad \operatorname{Re} \lambda \leqslant \lambda_K \end{cases} \\
&\geqslant \operatorname{dist}(\lambda, I_K) \|u\|^2;
\end{aligned}
$$

hence

$$\|R(\lambda, A_s)\|_{L^2, L^2} \leqslant \operatorname{dist}(\lambda, I_K)^{-1}. \tag{5.50}$$

$R(\lambda, \bar{A}_u)$ is a $(K-1)\times(K-1)$-matrix of the form

$$R(\lambda, \bar{A}_u) = \det(\bar{A}_u - \lambda)^{-1}\, p(\lambda, \bar{A}_u),$$

where $p(\lambda, \bar{A}_u)$ is a polynomial in λ (this follows easily from the inversion formula for matrices), and

$$\det(\bar{A}_u - \lambda) = \text{const} \cdot \prod_{1 \leqslant j < K} (\lambda - c_j). \tag{5.52}$$

Therefore $\|R(\lambda, \bar{A}_u)\|_{L^2, L^2}$ is $O(|\lambda - c_k|^{-1})$ is a neighbourhood of each $c_k \in \{c_j\}_{1 \leqslant j < K}$. For $|\lambda| \to \infty$ we write

$$\bar{A}_u - \lambda = -\lambda(1 - \lambda^{-1}\bar{A}_u) \tag{5.53}$$

and we see that $R(\lambda, \bar{A}_u)$ is $O(|\lambda|^{-1})$ for $|\lambda| \to \infty$, since $(1 - \lambda^{-1}\bar{A}_u)^{-1}$ can be expressed by a Neumann series in $\lambda^{-1}\bar{A}_u$ for $|\lambda| > \|\bar{A}_u\|_{L^2, L^2}$. We can conclude that

$$\|R(\lambda, \bar{A}_u)\|_{L^2, L^2} \leqslant M(\operatorname{dist}(\lambda, \operatorname{sp}(\bar{A}_u)))^{-1}, \tag{5.54}$$

where M is a constant independent of λ.

Obviously $\operatorname{sp}(\bar{A}_u) \subset I_K$, and since $\operatorname{sp}(\bar{A}_u)$ is bounded, $\|\lambda R(\lambda, \bar{A}_u)\|_{L^2, L^2}$ is $O(1)$ for $|\lambda| \to \infty$; moreover, $P_s K_v T'$ is a bounded operator, and the decomposition $f \to f_u + f_s$ is bounded and λ-independent. Then, from the form of $R(\lambda, A_1)$ (5.47)–(5.48), we find that

$$\begin{aligned}\|R(\lambda, A_1)\|^2_{L^2, L^2} &\leqslant \frac{M'^2}{\operatorname{dist}(\lambda, \operatorname{sp}(\bar{A}_u))^2} + \frac{1}{\operatorname{dist}(\lambda, I_K)^2}\\ &\leqslant \frac{M_1^2}{\operatorname{dist}(\lambda, I_K)^2},\end{aligned}$$

where M' and M_1 are positive constants.

Using Theorem 5.1 and Theorem 5.4 we find one of Lasiecka and Triggiani's results:

THEOREM 5.5. *Assume that*

1. *The Dirichlet traces $\{\gamma\varphi_j\}_{1 \leqslant j < K}$ of the Neumann eigenfunctions φ_j are linearly independent (but see also Remark 5.2),*

2. *The largest occurring multiplicity of the Neumann eigenvalues $\{\lambda_j\}_{1 \leqslant j < K}$ of A is N.*

Then there exists a finite-dimensional boundary condition

$$\upsilon u = T'u \qquad \text{on } \Gamma, \tag{5.55}$$

where

$$T'u = \sum_{j=1}^{N} (u \mid w_j)\, g_j, \qquad w_j \in X_u,\ g_j \in C^\infty(\Gamma)^m, \qquad j = 1, 2, ..., N, \tag{5.56}$$

such that the realization A_1 *of* A, *with domain*

$$D(A_1) = \{u \in H^{2m}(\Omega) \mid \upsilon u = T'u\} \tag{5.57}$$

is the infinitesimal generator of an analytic semigroup $e^{-A_1 t}$, $t \geqslant 0$, *on* $L^2(\Omega)$, *giving the solution to the Neumann boundary feedback parabolic system* (4.6) *as*

$$u(t, x) = e^{-A_1 t} u_0(x), \qquad x \in \Omega, \quad t \geqslant 0, \tag{5.58}$$

when $u_0 \in L^2(\Omega)$, *and such that the solution* (5.58) *satisfies the damping estimate*

$$\|u(t, \cdot)\| \leqslant M e^{-\lambda_K t} |u_0|, \qquad t \geqslant 0, \quad M > 0, \tag{5.59}$$

where λ_K *is the first positive Neumann eigenvalue of* A. *Moreover, the operators*

$$\cos(A_1^{1/2} t) \qquad \text{and} \qquad \sin(A_1^{1/2} t), \tag{5.60}$$

are well defined, and we can write the solution to the corresponding hyperbolic problem (4.5) *as*

$$u(t, x) = \cos(A_1^{1/2} t)\, u_0 + A_1^{-1/2} \sin(A_1^{1/2} t)\, u_1(x), \tag{5.61}$$

$x \in \Omega$, $t > 0$, *when* $u_0, u_1 \in L^2(\Omega)$.

Remark 5.6. We see that in the "decoupled" case, where $w_j \in X_u$, $j = 1, ..., K-1$, the estimate (5.59) holds. When $P_s w_j \neq 0$, the damping coefficient λ_K must be substituted by $\lambda_K - \varepsilon$, as we shall see later.

Now it is straightforward to extend the theory to include more general cases, where $P_s w_j \neq 0$. The operator T' considered above can be written

$$T'u = \sum_{j=1}^{N} (u \mid P_u w_j)\, g_j, \tag{5.62}$$

so if we let the w_j be arbitrary and define the operator T'' as

$$T''u = \sum_{j=1}^{N} (u \mid P_s w_j)\, g_j, \tag{5.63}$$

we see that the decoupled case considered above corresponds to the case where $T''=0$. Now, in order to simplify the notation, let us denote by T and T_1 the operators defined by

$$Tu = \upsilon u - T'u \tag{5.64}$$

and

$$T_1 u = \upsilon u - T'u - T''u. \tag{5.65}$$

Then T_1 defines a normal boundary condition $T_1 u = 0$' see [P1] (in the sense of Grubb [G1]), just as T did; hence the operator realization A_{T_1} of A, with domain

$$D(A_{T_1}) = \{u \in H^{2m}(\Omega) \mid T_1 u = 0\}, \tag{5.66}$$

is a densely defined, closed operator in $L^2(\Omega)$. Here T_1 can be regarded as a perturbation of the trace operator $T = \gamma - T'$ considered above.

Let K_T be the Poisson solution operator defined by $u = K_T \varphi$, where u is the solution of

$$\begin{aligned} Au &= 0 \qquad \text{in } \Omega, \\ Tu &= \varphi \qquad \text{on } \Gamma. \end{aligned} \tag{5.67}$$

We assume in the following that T (i.e., the sets $\{P_u w_j\}_{1 \leqslant j < K}$ and $\{h_j\}_{1 \leqslant j < K}$) are chosen such that the conclusions of Theorem 5.5 are valid, and then we study A_{T_1}.

We use here that A_1 is chosen to be bijective from $D(A_1)$ to $L^2(\Omega)$, which implies that K_T is well defined. Observe also the estimate

$$\begin{aligned} \|K_T T''u\| &= \left\| \sum_{j=1}^{N} (u \mid P_s w_j)\, K_T g_j \right\| \\ &\leqslant \|u\| \sum_{j=1}^{N} \|P_s w_j\| \, \|K_T g_j\|, \end{aligned} \tag{5.70}$$

where $\|K_T g_j\|$ depends only the sets $\{P_u w_j\}_{1 \leqslant j < K}$ and $\{g_j\}_{1 \leqslant j < K}$.

LEMMA 5.7. *Assume that the sets $\{P_u w_j\}_{1 \leqslant j < K}$ and $\{g_j\}_{1 \leqslant j < K}$ are chosen such that the conclusions of Theorem 5.4 are valid. Then there exists a constant $r_1 > 0$, such that for $\|P_s w_j\| < r_1$, $j = 1, 2, ..., N$, the operator*

$$1 - K_T T'' \tag{5.71}$$

is a homeomorphism in $L^2(\Omega)$ and in $H^{2m}(\Omega)$ and, in particular, defines a bijection

$$1 - K_T T'' \colon D(A_{T_1}) \xrightarrow{\sim} D(A_1). \tag{5.72}$$

Moreover, when $u \in D(A_{T_1})$ *and* $v = (1 - K_T T'')u$, *then* $Au = Av$; *in fact one has the factorization*

$$A_{T_1} = A_1(1 - K_T T''). \tag{5.73}$$

Proof. By (5.70) it is possible to chose $r_1 > 0$ such that for $\|P_s w_j\| < r_1$, $j = 1, 2, ..., N$, we have $\|K_T T''\|_{L^2, L^2} \leq \frac{1}{2}$. Now $1 - K_T T''$ is a bounded operator in $L^2(\Omega)$ and is inverted by a Neumann series

$$(1 - K_T T'')^{-1} = \sum_{m=0}^{\infty} (K_T T'')^m, \tag{5.74}$$

converging in the operator norm in $L^2(\Omega)$. Thus $1 - K_T T''$ is a homeomorphism of $L^2(\Omega)$ onto itself.

Since K_T has range in $H^{2m}(\Omega)$, we see that $1 - K_T T''$ is likewise a homeomorphism of $H^{2m}(\Omega)$ onto itself, and, since

$$T(1 - K_T T'')u = Tu - T''u = T_1 u, \tag{5.75}$$

we see that $u \in D(A_{T_1})$ if and only if $v = (1 - K_T T'')u \in D(A_1)$, so $1 - K_T T''$ defines a bijection of $D(A_{T_1})$ onto $D(A_1)$. The last observation follows from the fact that $AK_T = 0$.

We will now study the resolvent $R(\lambda, A_{T_1})$ of A_{T_1}, and we start out with the equation

$$\begin{aligned} (A - \lambda)u &= f \qquad \text{in } \Omega, \\ T_1 u &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{5.76}$$

Using (5.72) with $v = (1 - K_T T'')u$ we obtain

$$\begin{aligned} (A - \lambda)(v + K_T T''u) &= f \qquad \text{in } \Omega, \\ Tv &= 0 \qquad \text{on } \Gamma, \end{aligned} \tag{5.77}$$

so that

$$\begin{aligned} (A - \lambda)v &= f + \lambda K_T T''u \qquad \text{in } \Omega, \\ Tv &= 0 \qquad \text{on } \Gamma; \end{aligned} \tag{5.78}$$

i.e., if λ is in the resolvent set of A_1,

$$\begin{aligned} v &= R(\lambda, A_1)(f + \lambda K_T T''u) \qquad \text{in } \Omega, \\ Tv &= 0 \qquad \text{on } \Gamma. \end{aligned} \tag{5.79}$$

For u this gives

$$u - K_T T''u = v = R(\lambda, A_1)(f + \lambda K_T T''u), \tag{5.80}$$

so that

$$(1-(1+\lambda R(\lambda, A_1))\, K_T T'')u = R(\lambda, A_1)\, f. \tag{5.81}$$

Let us denote, for $\varepsilon > 0$, $0 < \theta < \pi/2$ the obtuse sector (disjoint from I_K)

$$W_{\lambda_K, \varepsilon, \theta} = \left\{ z \in \mathbb{C} \mid z = (\lambda_K - \varepsilon) + re^{i\omega},\ r \geqslant 0, \frac{\pi}{2} - \theta < \omega < \frac{3\pi}{2} + \theta \right\}.$$

From the estimate (5.49) we see that $\lambda R(\lambda, A_1)$ is bounded on $W_{\lambda_K, \varepsilon, \theta}$, and hence for any $\varepsilon > 0$, any $\theta \in]0, \pi/2[$, there exists by (5.70) a constant $r > 0$ such that for $w_j \in L^2(\Omega)$, satisfying $\|P_s w_j\|_0 < r \leqslant r_1$, $j = 1, 2, ..., N$, we have

$$\|(1+\lambda R(\lambda, A_1))\, K_T T''\|_{L^2, L^2} \leqslant \tfrac{1}{2} \tag{5.82}$$

for all $\lambda \in W_{\lambda_K, \varepsilon, \theta}$.

With the w_j, $j = 1, 2, ..., N$, chosen in this way, the resolvent $R(\lambda, A_{T_1})$ of A_{T_1} is a well-defined, bounded operator in $L^2(\Omega)$ for $\lambda \in W_{\lambda_K, \varepsilon, \theta}$, given by (see (5.81))

$$\begin{aligned} R(\lambda, A_{T_1}) &= (1-(1+\lambda R(\lambda, A_1))\, K_T T'')^{-1} R(\lambda, A_1) \\ &= \sum_{m=0}^{\infty} ((1+\lambda R(\lambda, A_1))\, K_T T'')^m R(\lambda, A_1) \end{aligned} \tag{5.83}$$

and satisfying the estimate

$$\|R(\lambda, A_{T_1})\|_{L^2, L^2} \leqslant \frac{c_1}{|\lambda - \lambda_K|}. \tag{5.84}$$

Here $c_1 > 0$ is a constant, independent of λ.

Altogether, we have obtained the following results:

THEOREM 5.8. *Let $\varepsilon > 0$ be given, and assume that the sets $\{P_u w_j\}_{1 \leqslant j < K}$ and $\{g_j\}_{1 \leqslant j < K}$ are chosen such that the conclusions of Theorem 5.5 are valid. The finite-dimensional Neumann boundary feedback*

$$\gamma u = T'u + T''u = \sum_{j=1}^{N} (u \mid w_j)\, g_j \tag{5.85}$$

defines a realization A_{T_1} of A, with domain

$$D(A_{T_1}) = \{u \in H^{2m}(\Omega) \mid T_1 u = 0\}, \tag{5.86}$$

where T_1 is the trace operator defined by

$$T_1 = \upsilon - T' - T'', \tag{5.87}$$

and there exists a constant $r>0$, such that for arbitrary choises of $P_s w_j$, with $\|P_s w_j\|<r$, A_{T_1} is the infinitesimal generator of an analytic semigroup $e^{-A_{T_1}t}$, $t \geqslant 0$, on $L^2(\Omega)$, giving the solution to the Neumann boundary feedback control system

$$\begin{aligned} \partial_t u + Au &= 0 && \textit{in } \Omega \quad \textit{for } t>0, \\ \nu u &= \sum_{j=1}^{N} (u \mid w_j)\, g_j && \textit{on } \Gamma \quad \textit{for } t>0, \\ u &= u_0 && \textit{in } \Omega, \quad \textit{at } t=0, \end{aligned} \tag{5.88}$$

as

$$u(t,x) = e^{-A_{T_1}t} u_0(x), \qquad t \geqslant 0\ x \in \Omega,\ u_0 \in L^2(\Omega) \tag{5.89}$$

where the solution (5.89) satisfies

$$\|u(t,\cdot)\| \leqslant M_{T_1} e^{-(\lambda_K - \varepsilon)t} \|u_0\|, \qquad t \geqslant 0. \tag{5.90}$$

Here λ_K is the first positive Neumann eigenvalue of A, and M_{T_1} is a constant >0.

We will now use the factorization (5.73) in the investigation of the hyperbolic problem for A_{T_1}. The boundary value problem

$$\partial_t^2 u + A_{T_1} u = 0, \qquad u \in D(A_{T_1}) \tag{5.91}$$

transforms by (5.73) into

$$(1 - K_T T'')^{-1}\, \partial_t^2 v + A_1 v = 0, \qquad v \in D(A_1), \tag{5.92}$$

where $v = (1 - K_T T'')u$.

Composing with $(1 - K_T T'')$ from the left in (5.92) we find

$$\partial_t^2 v + (1 - K_T T'')\, A_1 v = 0, \qquad v \in D(A_1). \tag{5.93}$$

If we now, moreover, impose on the w_j to satisfy $P_s w_j \in D(A_1^*)$, $j = 1, 2, ..., N$, then for $v \in D(A_1)$,

$$\begin{aligned} \|K_T T'' A_1 v\| &= \left\| \sum_{j=1}^{N} (A_1 v \mid P_s w_j)\, K_T g_j \right\| \\ &= \left\| \sum_{j=1}^{N} (v \mid A_1^* P_s w_j)\, K_T g_j \right\| \\ &\leqslant \|v\| \sum_{j=1}^{N} \|A_1^* P_s w_j\|\, \|K_T g_j\|. \end{aligned}$$

This shows that $K_T T'' A_1$ acts like an L^2-bounded operator on $D(A_1)$, when $P_s w_j \in D(A_1^*)$, $j = 1, 2, ..., N$.

Therefore, (5.93) (and with it (5.91) in a related sense) is simply a *bounded* perturbation of the equation

$$\begin{aligned} &\partial_t^2 v + Av = 0 && \text{in } \Omega && \text{for} \quad t>0,\\ &Tv = 0 && \text{on } \Gamma && \text{for} \quad t>0,\\ &v = v_0 && \text{in } \Omega, && \text{at} \quad t=0,\\ &\partial_t v = v_1 && \text{in } \Omega, && \text{at} \quad t=0, \end{aligned} \tag{5.94}$$

treated in Theorem 5.5. Since the spectrum of A_1 is assumed to be contained in $\mathbb{R}_+$, we find from standard bounded perturbation theory (see, e.g., Sova [S1] and Fattorini [F1]) that the operators

$$\cos((A_T - K_T T'' A_1)^{1/2} t) \tag{5.95}$$

$$\sin((A_T - K_T T'' A_1)^{1/2} t) \tag{5.96}$$

are well-defined, bounded operators in $L^2(\Omega)$ for $P_s w_j \in D(A_1^*)$, $j = 1, 2, ..., N$, and $t > 0$. From Theorem 1.6.11 in Grubb [G1] we find that

$$D(A_1^*) = \{u \in H^{2m}(\Omega) \mid \dot{I}^x(\mathscr{A}^{10*}\nu u + \mathscr{A}^{00*}\gamma u) = 0\}, \tag{5.97}$$

where $\dot{I}^x$ is the "reflection" of the index set, replacing $\{k\}_{0 \leqslant k < m}$ by $\{2m-k-1\}_{0 \leqslant k < m}$ and $\mathscr{A}^{10}$ and $\mathscr{A}^{00}$ are the $m \times m$-matrix differential operators, appearing in Greens formula for $\mathscr{A}$. We see from Ex. 1.6.14 in Grubb [G1] that the action of the realization A_1^* is of the form $A + G$, for a certain singular Green operator G of finite rank.

We have thus obtained

THEOREM 5.9. *Let the set* $\{w_j, g_j\}_{1 \leqslant j \leqslant N}$ *be chosen according to Theorem* 5.8 *and assume, furthermore, that* $P_s w_j$, $j = 1, 2, ..., N$, *are chosen in* $D(A_1^*)$. *Then the operators*

$$C(t) = \cos((A_1 - K_T T'' A_1)^{1/2} t) \tag{5.98}$$

and

$$S(t) = (A_1 - K_T T'' A_1)^{-1/2} \sin((A_1 - K_T T'' A_1)^{1/2} t) \tag{5.99}$$

on $L^2(\Omega)$, *are well defined for* $t \in \mathbb{R}$, *giving the solution to the system*

$$\begin{aligned} &\partial_t^2 v + (1 - K_T T'')\, Av = 0 && \text{in } \Omega \quad \text{for } t>0,\\ &\nu v = \sum_{j=1}^{N} (v \mid P_u w_j)\, g_j && \text{on } \Gamma \quad \text{for } t>0,\\ &v = v_0 && \text{in } \Omega, \quad \text{at } t=0,\\ &\partial_t v = v_1 && \text{in } \Omega, \quad \text{at } t=0, \end{aligned} \tag{5.100}$$

as

$$v(t, x) = C(t)\, v_0(x) + S(t)\, v_1(x), \qquad x \in \Omega,\ t > 0,\ v_0, v_1 \in L^2(\Omega). \tag{5.101}$$

The solution to the original system

$$\begin{aligned} \partial_t^2 u + Au &= 0 && \text{in } \Omega \quad \text{for } t > 0, \\ \nu u &= \sum_{j=1}^{N} (u \mid w_j)\, g_j && \text{on } \Gamma \quad \text{for } t > 0, \\ u &= u_0 && \text{in } \Omega, \quad \text{at } t = 0, \\ \partial_t u &= u_1 && \text{in } \Omega, \quad \text{at } t = 0, \end{aligned} \tag{5.102}$$

is then by (5.73)

$$\begin{aligned} u(t, x) = {} & (1 - K_T T'')^{-1} C(t)(1 - K_T T'')\, u_0(x) \\ & + (1 - K_T T'')^{-1} S(t)(1 - K_T T'')\, u_1(x), \\ & x \in \Omega,\ t > 0,\ u_0, u_1 \in L^2(\Omega). \end{aligned} \tag{5.103}$$

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